S.-T. Yau College Student Mathematics Contests 2024

Algebra, Number Theory and Combinatiorics

Team

Problem 1. *Find all the diagonalisable invertible $n \times n$ complex matrices A with real eigenvalues such that*

$$A^T - I = A + A^{-1}.$$

Problem 2. *For any $f \in \mathbb{Q}[x]$, we let $rad(f)$ be the product of all prime polynomials dividing f. (This is unique up to nonzero scalar multiple.)*

Let $P, Q, R \in \mathbb{Z}[x]$ be nonconstant relatively-prime polynomials that satisfy $P + Q = R$. Prove that

$$\deg R < \deg \ rad(PQR).$$

Problem 3. *(1). Consider the profinite group $A = \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$. Show that a subgroup of A is open if and only if it is of finite index.*

(2). Let p be a prime number, $\mathbb{N}$ be the set of natural numbers, $B = \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$, equipped with the product topology. Does the analogous equivalence in (1) above holds for B? Justify your assertion.